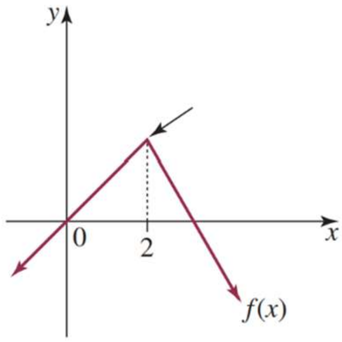
## Conditions for Differentiability

The gradient of a function exists where it's **smooth** and **continuous**.

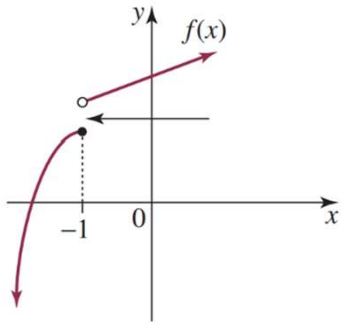
A function is smooth if its graph has no sharp points (cusps).

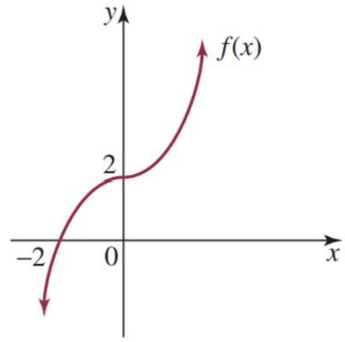
It's continuous if there are no breaks, holes, gaps, jumps, or asymptotes on its graph.




The graph is NOT smooth at $x = 2$, so the gradient does NOT exist at this point. The gradient exists for $x \in \mathbb{R} \setminus \{2\}$

The graph has a break at $x = -1$, so the gradient does NOT exist at this point. The gradient exists for $x \in \mathbb{R} \setminus \{-1\}$



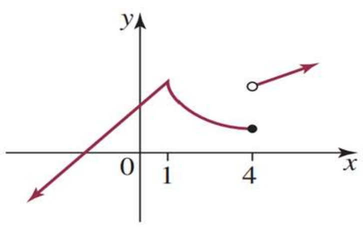

The graph is smooth and continuous for all values of $x$. The gradient exists for $x \in \mathbb{R}$

## Limits and Continuity

A function $f$ is continuous at the point $x = a$ if the following 3 conditions are met:

- $f(x)$ is defined at $x = a$
- $\lim_{x \to a^-} f(x) = f(a)$
- $\lim_{x \to a^+} f(x) = f(a)$

A function $f$ is smooth at the point $x = a$ if its derivative exists and is continuous at $x = a$.




The gradient exists for $x \in \mathbb{R} \setminus \{1,4\}$

Let $f(x) = \begin{cases} \tan\left(\frac{x}{2}\right), & 4 \le x < 2\pi \\ \sin(ax), & 2\pi \le x \le 8 \end{cases}$

Find the value of $a$ for which $f(x)$ is continuous and smooth at $x = 2\pi$.

$f$ is continuous at $x = 2\pi$ when

$$\tan\left(\frac{2\pi}{2}\right) = \sin(2a\pi)$$

$f$ is smooth at $x = 2\pi$ when

$$\frac{d}{dx}\tan\left(\frac{2\pi}{2}\right) = \frac{d}{dx}\sin(2a\pi)$$

Solve simultaneously via CAS

$\therefore a = -\frac{1}{2}$